

# A VALUATION THEORETIC CHARACTERIZATION OF RECURSIVELY SATURATED REAL CLOSED FIELDS

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**ABSTRACT.** We give a valuation theoretic characterization for a real closed field to be recursively saturated. This builds on work in [KKMZ02], where the authors gave such a characterization for  $\kappa$ -saturation, for a cardinal  $\kappa \geq \aleph_0$ . Our result extends the characterization of Harnik and Ressayre [HR] for a divisible ordered abelian group to be recursively saturated.

## 1. INTRODUCTION

Recursive saturation was introduced by Barwise and Schlipf in [BS76].

**Definition 1.1.** ([BS76]) A structure  $\mathcal{M}$  for a computable language  $L$  is *recursively saturated* if given a computable set of  $L$ -formulas  $\tau(x, \bar{v})$  and a tuple  $\bar{a}$  in  $\mathcal{M}$  appropriate to substitute for  $\bar{v}$  such that every finite subset of  $\tau(x, \bar{a})$  is satisfied in  $\mathcal{M}$ , then the whole  $\tau(x, \bar{a})$  is satisfied in  $\mathcal{M}$ .

In [DKS10] a characterization of countable recursively saturated real closed fields was obtained in terms of their integer parts.  $\kappa$ -saturation (for an arbitrary infinite cardinal  $\kappa$ ) has been investigated in terms of valuation theory for divisible ordered abelian groups in [Ku90], real closed fields in [KKMZ02], and more generally for o-minimal expansions of real closed fields in [CDK]. In this paper we extend the above valuation theoretical characterizations to recursively saturated real closed fields, thereby extending results of [HR] for divisible ordered abelian groups.

## 2. PRELIMINARIES

**2.1. Scott sets.** A subset  $\mathcal{T} \subset 2^{<\omega}$  is a *tree* if every substring of an element of  $\mathcal{T}$  is also an element of  $\mathcal{T}$ . If  $\sigma, \tau \in 2^{<\omega}$ , we let  $\sigma \prec \tau$  denote that  $\sigma$  is a substring of  $\tau$ . A sequence  $f \in 2^\omega$  is a *path* through a tree  $\mathcal{T}$  if for all  $\sigma \in 2^{<\omega}$  with  $\sigma \prec f$ , the string  $\sigma$  is an element of  $\mathcal{T}$ . For any string  $\sigma \in 2^{<\omega}$ , the length of  $\sigma$ , denoted by  $length(\sigma)$ , is the unique  $n \in \omega$  satisfying  $\sigma \in 2^n$ .

**Definition 2.1.** A nonempty set  $S \subset \mathbb{R}$  is a *Scott set* if

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- (1)  $S$  is computably closed, i.e., if  $r_1, \dots, r_n \in S$  and  $r \in \mathbb{R}$  is computable from (the Turing join of)  $r_1, \dots, r_n$ , then  $r \in S$ .
- (2) If an infinite tree  $\mathcal{T} \subset 2^{<\omega}$  is computable in some  $r \in S$ , then  $\mathcal{T}$  has a path that is computable in some  $r' \in S$ .

**Fact 2.2.** Any Scott set  $S$  is an archimedean real closed field.

**2.2. Some valuation theoretic notions.** We summarize the required background (see [Ku00] and [Ku90]). Let  $(G, +, 0, <)$  be a divisible ordered abelian group. Given  $A \subset G$ , we let  $\langle A \rangle_{\mathbb{Q}}$  denote the smallest divisible ordered subgroup of  $G$  containing  $A$ . For any  $x \in G$  let  $|x| = \max\{x, -x\}$ . For non-zero  $x, y \in G$  we define  $x \sim y$  if there exists  $n \in \mathbb{N}$  such that  $n|x| \geq |y|$  and  $n|y| \geq |x|$ . We write  $x << y$  if  $n|x| < |y|$  for all  $n \in \mathbb{N}$ . Clearly,  $\sim$  is an equivalence relation, and we let  $[x]$  denote the equivalence class of any non-zero  $x \in G$ . Let  $\Gamma := G - \{0\} / \sim = \{[x] : x \in G - \{0\}\}$ . We can define an order on  $<_{\Gamma}$  in terms of  $<<$  as follows,  $[y] <_{\Gamma} [x]$  if  $x << y$  (notice the reversed order). Given a linear ordering  $(A, <)$  and  $A_1, A_2 \subset A$ , we use the notation  $A_1 < A_2$  to indicate that  $a_1 < a_2$  for all  $a_1 \in A_1$  and  $a_2 \in A_2$ .

**Fact 2.3.** (a)  $\Gamma$  is a totally ordered set under  $<_{\Gamma}$ , and we will refer to it as the value set of  $G$ .

(b) The map

$$\begin{aligned} v: G &\longrightarrow \Gamma \cup \{\infty\} \\ 0 &\mapsto \infty \\ x &\mapsto [x] \quad (\text{if } x \neq 0) \end{aligned}$$

is a valuation on  $G$  as a  $\mathbb{Z}$ -module, i.e. for every  $x, y \in G$ :

$v(x) = \infty$  if and only if  $x = 0$ ,  $v(nx) = v(x)$  for all  $n \in \mathbb{Z}$ ,  $n \neq 0$ , and  $v(x + y) \geq \min\{v(x), v(y)\}$ .

(c) For every  $\gamma \in \Gamma$  the Archimedean component associated to  $\gamma$  is the maximal Archimedean subgroup of  $G$  containing some  $x \in \gamma$ . We denote it by  $A_{\gamma}$ . For each  $\gamma$ ,  $A_{\gamma}$  is isomorphic to an ordered subgroup of  $(\mathbb{R}, +, 0, <)$ . Furthermore, we can calculate the isomorphism type of  $A_{\gamma}$  in terms of any  $x \in \gamma$ . Given  $x, y \in \gamma$ , we let  $\frac{y}{x} = \sup\{r \in \mathbb{Q} \mid rx < y\}$ , and let  $A_{\gamma, x} = \{\frac{y}{x} \mid y \in \gamma\} \cup \{0\}$ . Then,  $A_{\gamma} \cong A_{\gamma, x}$ .

(d) Since  $G$  is a divisible abelian group, we may view  $G$  as a vector space over  $\mathbb{Q}$ . We focus on the case where  $G$  is finite dimensional as a vector space, as we will use the next notion of valuation independence exclusively in that context. A set  $\{g_1, \dots, g_n\} \subset G$  is called *valuation independent* if for all  $q_1, \dots, q_n \in \mathbb{Q}$ ,

$$v(q_1 g_1 + \dots + q_n g_n) = \min\{v(g_i) \mid q_i \neq 0\}$$

A basis  $\{g_1, \dots, g_n\}$  for  $G$  is called a *valuation basis* if it is a valuation independent set. A theorem by Brown [Br] states that every vector space of countable dimension with a valuation admits a valuation basis.

**Definition 2.4.** Let  $\lambda$  be an infinite ordinal. A sequence  $(a_\rho)_{\rho < \lambda}$  contained in  $G$  is said to be *pseudo Cauchy* (or *pseudo convergent*) if for every  $\rho < \sigma < \tau < \lambda$  we have

$$v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma).$$

**Fact 2.5.** If  $(a_\rho)_{\rho < \lambda}$  is pseudo Cauchy sequence then for all  $\rho < \sigma < \lambda$  we have

$$v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho).$$

**Definition 2.6.** Let  $(a_\rho)_{\rho < \lambda}$  be a pseudo Cauchy sequence in  $G$ . We say that  $x \in G$  is a *pseudo limit* of  $S$  if

$$v(x - a_\rho) = v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho) \quad \text{for all } \rho < \sigma < \lambda.$$

If  $(R, +, \cdot, 0, 1, <)$  is an ordered field then it has a natural valuation  $v$ , that is the natural valuation associated with the ordered abelian group  $(R, +, 0, <)$ . We will denote by  $G$  the value group of  $R$  with respect to  $v$ , i.e.,  $G = v(R)$ . If  $(R, +, \cdot, 0, 1, <)$  is a real closed field then  $G$  is divisible, and we will refer to the linear dimension of  $G$  as a  $\mathbb{Q}$ -vector space as the *rational rank* of  $G$ , denoted  $\text{rk}(G)$ . For the natural valuation on  $R$ , we use the notations  $\mathcal{O}_R = \{r \in R : v(r) \geq 0\}$  and  $\mu_R = \{r \in R : v(r) > 0\}$  for the valuation ring and the valuation ideal, respectively. The residue field  $k$  is the quotient  $\mathcal{O}_R/\mu_R$ , and we recall that it is isomorphic to a unique subfield of  $\mathbb{R}$ . When convenient, we identify  $k$  with this unique subfield of  $\mathbb{R}$ . Given any  $a \in R$  with  $v(a) \geq 0$ , we denote the residue of  $a$  by  $\bar{a} \in k$ , i.e.,  $\bar{a}$  is the unique element in  $k$  such that  $v(a - \bar{a}) > 0$ . Notice that in the case of ordered fields there is a unique archimedean component up to isomorphism, and if the field is real closed, the archimedean component is the residue field.

If  $R$  is a real closed field, given  $X \subset R$ , we let  $RC(\mathbb{Q}(X))$  denote the real closure of  $\mathbb{Q}(X)$  in  $R$ . A notion of pseudo Cauchy sequence is easily extended to any ordered field as in the case of ordered abelian groups.

### 3. BACKGROUND ON $\kappa$ -SATURATED STRUCTURES

We now recall the characterization of  $\aleph_\alpha$ -saturation for divisible ordered abelian groups given in [Ku90]. We need the notion of  $\eta_\alpha$ -sets (see [R]). An  $\eta_\alpha$ -set is a linear ordering  $(A, <)$  such that, whenever  $A_1, A_2 \subset A$  have cardinality less than  $\aleph_\alpha$  and  $A_1 < A_2$ , then there is an  $a \in A$  such that  $A_1 < a < A_2$ . Observe that an  $\eta_0$ -set is simply a dense linear ordering without endpoints.

**Theorem 3.1.** [Ku90] *Let  $G$  be a divisible ordered abelian group, and let  $\aleph_\alpha \geq \aleph_0$ . Then  $G$  is  $\aleph_\alpha$ -saturated in the language of ordered groups if and only*

- (i) *the value set of  $G$  is an  $\eta_\alpha$ -set,*
- (ii) *all the archimedean components of  $G$  are isomorphic to  $\mathbb{R}$ , and*

- (iii) every pseudo Cauchy sequence in a divisible subgroup of  $G$  with a value set of cardinality less than  $\aleph_\alpha$  has a pseudo limit in  $G$ .

Notice that in the case of  $\aleph_0$ -saturation the necessary and sufficient conditions reduce only to (1) and (2).

The following characterization of  $\aleph_\alpha$ -saturated real closed fields was obtained in [KKMZ02].

**Theorem 3.2.** [KKMZ02, Theorem 6.2] *Let  $R$  be a real closed field,  $v$  its natural valuation,  $G$  its value group and  $k$  its residue field. Let  $\aleph_\alpha \geq \aleph_0$ . Then  $R$  is  $\aleph_\alpha$ -saturated in the language of ordered fields if and only if*

- (i)  $G$  is  $\aleph_\alpha$ -saturated,
- (ii)  $k \cong \mathbb{R}$ ,
- (iii) every pseudo Cauchy sequence in a subfield of  $R$  of absolute transcendence degree less than  $\aleph_\alpha$  has a pseudo limit in  $R$ .

In the proof of Theorem 3.2 the *dimension inequality* (see [EP]) is crucially used in the case of  $\aleph_0$ -saturation. This says that the rational rank of the value group of a finite transcendental extension of a real closed field is bounded by the transcendence degree of the extension.

#### 4. RECURSIVELY SATURATED DIVISIBLE ORDERED ABELIAN GROUPS

Harnik and Ressayre state the following result in [HR] and sketch a proof just for the necessity of condition (ii). We include here a complete proof.

**Theorem 4.1.** *Let  $G$  be a divisible ordered abelian group. Then  $G$  is recursively saturated in the language of ordered groups if and only if*

- (i) the value set of  $G$  is an  $\eta_0$ -set, and
- (ii) all archimedean components of  $G$  equal a common Scott set  $S$ .

*Proof.* Suppose  $G$  is recursively saturated. We show that (i) and (ii) hold.

- (i) Let  $g, g' \in G$  such that  $g, g' > 0$  and  $v(g) < v(g')$ . The partial type

$$\beta(x, g, g') = \{ng' < x \mid n \in \mathbb{N}\} \cup \{x < ng \mid n \in \mathbb{N}\}$$

is computable and finitely satisfiable (since  $v(g) < v(g')$ ). By recursive saturation, there is some  $h \in G$  such that  $\beta(h, g, g')$  holds in  $G$ , and  $v(g) < v(h) < v(g')$ .

- (ii) We first show that  $A_{[g],g} = A_{[g'],g'}$  for all nonzero  $g, g' \in G$ . Let  $r \in A_{[g],g}$ . Let  $\delta(x, y, g, g')$  be the partial type consisting of all formulas with  $q, q' \in \mathbb{Q}$  and  $q < q'$  of the form

$$qq < y < q'g \rightarrow qq' < x < q'g'.$$

Since  $r \in A_{[g],g}$ , there exists some  $h \in G$  so that  $\frac{h}{g} = r$ . The set of formulas  $\delta(x, h, g, g')$  is computable and finitely satisfiable in  $G$  since

$G$  is divisible, so there is some  $h' \in G$  so that  $\delta(h', h, g, g')$  holds in  $G$ . Then  $\frac{h'}{g} = r$ , so  $r \in A_{[g'], g'}$ . We have that  $A_{[g], g} = A_{[g'], g'}$  by a symmetric argument. Hence, it is well defined to refer to  $A_{[g], g}$  simply as  $A_{[g]}$ .

Let  $g \in G$ . We show that  $A_{[g]}$  is a Scott set. Suppose  $r_1, \dots, r_n \in A_{[g]}$  and  $r \in \mathbb{R}$  is computable in  $r_1, \dots, r_n$  via some Turing reduction  $\Psi$ . Take  $g_i \in G$  such that  $r_i = \frac{g_i}{g}$  for  $1 \leq i \leq n$ . For each  $n$ -tuple of pairs of rationals  $(q_i < q'_i)_{i=1}^n$ , each stage  $s \in \mathbb{N}$ , and another pair of rationals  $\hat{p} < \hat{p}'$ , compute whether  $\Psi$ , using only the knowledge that arbitrary input reals  $\tilde{r}_1, \dots, \tilde{r}_n$  satisfy  $q_i < \tilde{r}_i < q'_i$  for  $1 \leq i \leq n$ , halts in  $s$  steps and outputs a real between  $p$  and  $p'$ . If  $\Psi$  halts in this situation, enumerate the formula

$$(q_1 g < g_1 < q'_1 g \wedge \dots \wedge q_n g < g_n < q'_n g) \rightarrow pg < x < p'g$$

into the partial type  $\zeta(x, g_1, \dots, g_n)$ . The partial type  $\zeta(x, g_1, \dots, g_n)$  is computably enumerable and finitely satisfiable since  $G$  is divisible. By recursive saturation, there is some  $h \in G$  such that  $\zeta(h, g_1, \dots, g_n)$  holds in  $G$ . Since  $\Psi$  in fact computes  $r$  from  $r_1, \dots, r_n$ , we have that  $r = \frac{h}{g} \in A_{[g]}$ , as desired.

Let  $\mathcal{T} \subset 2^{<\omega}$  be an infinite tree computable in some  $r \in A_{[g]}$ . We show that  $\mathcal{T}$  has a path computable in some  $r' \in A_{[g]}$ . Fix a computable function

$$\begin{aligned} (1) \quad f : 2^{<\omega} &\longrightarrow \{(a, b) \in \mathbb{Q}^2 \mid a < b\} \\ (2) \quad \sigma &\longrightarrow I_\sigma = (a_\sigma, b_\sigma) \end{aligned}$$

satisfying the following properties for  $\sigma, \tau \in 2^{<\omega}$ :

- (a)  $b_\sigma - a_\sigma = 2^{-\text{length}(\sigma)}$ ,
- (b)  $I_\sigma \cap I_\tau = \emptyset$  if  $\sigma \not\prec \tau$  and  $\sigma \not\succ \tau$ , and
- (c)  $I_\sigma \subset I_\tau$  if  $\sigma \succ \tau$ .

Let  $\mathcal{T} \subset 2^{<\omega}$  be a tree that is computable from some  $r \in A_{[g]}$  via a Turing reduction  $\Lambda$ . Let  $\mathcal{T}(k)$  be the set of nodes in  $\mathcal{T}$  of length  $k \in \omega$ , and let  $I_{\mathcal{T}(k)} = \cup_{\sigma \in \mathcal{T}(k)} I_\sigma$ . Fix some nonzero  $g \in G$ . Since  $r \in A_{[g]}$ , there is some  $h \in G$  such that  $r = \frac{h}{g}$ .

For each pair of rationals  $q < q'$  and  $s, k \in \mathbb{N}$ , compute whether Turing reduction  $\Lambda$ , using only the information that an arbitrary input real  $\tilde{r}$  satisfies  $q < \tilde{r} < q'$ , halts in  $s$  steps and outputs a (finite) set of nodes  $A \subset 2^k$ . If  $\Lambda$  halts in this situation, enumerate the formula

$$qg < h < q'g \rightarrow \frac{x}{g} \in \cup_{\sigma \in A} I_\sigma$$

into the partial type  $\kappa(x, h, g)$ . Note that  $\frac{x}{g} \in \cup_{\sigma \in A} I_\sigma$  can be expressed as a quantifier free formula in the language of divisible ordered groups. The partial type  $\kappa(x, h, g)$  is computably enumerable and finitely satisfiable since  $G$  is divisible. By recursive saturation, there is some  $h' \in G$  such that  $\kappa(h', h, g)$  holds in  $G$ . Since  $\Lambda$  computes  $\mathcal{T}$  from  $r$ , for each

$k \in \mathbb{N}$ , there is some stage  $s \in \mathbb{N}$  such that  $\Lambda$  computes  $\mathcal{T}(k)$  from  $r$ . Hence, for each  $k \in \mathbb{N}$ , there is some  $q, q' \in \mathbb{Q}$  with  $r \in (q, q')$  such that the formula

$$qg < h < q'g \rightarrow \frac{x}{g} \in \cup_{\sigma \in \mathcal{T}(k)} I_\sigma$$

is in  $\kappa(x, h, g)$ . Since  $r = \frac{h}{g}$ , the real  $r' = \frac{h'}{g} \in A_{[g]}$  is in  $\cup_{\sigma \in \mathcal{T}(k)} I_\sigma$  for all  $k \in \mathbb{N}$ . Then, the set  $\mathcal{P} = \{\sigma \in \mathcal{T} \mid r' \in I_\sigma\}$  is a path through  $\mathcal{T}$ . Moreover,  $\mathcal{P}$  is computable from  $r'$  since the assignment function  $f$  of nodes to intervals is a computable function and if  $r' \in I_\tau$  for some  $\tau \in \mathcal{T}$ , then  $r'$  is in  $I_{\tau_0}$  or  $I_{\tau_1}$  and it is computable to determine which one. We have finished showing that  $A_{[g]}$  is a Scott set, and the necessity portion of the theorem.

Now, let  $G$  be a divisible ordered abelian group. We show that if  $G$  satisfies (i) and (ii), then  $G$  is recursively saturated. Let  $S \subset \mathbb{R}$  be the Scott set such that  $S = A_{[g],g}$  for all  $g \in G$  by (ii). The proof follows the structure of the proof of Theorem 3.1 for the case of  $\aleph_0$ -saturation. The proof differs, however, in that its most interesting aspect is finding (using  $S$ ) a complete extension of the given computable partial type that is finitely satisfiable with the given parameters.

Let  $\bar{g} = (g_1, \dots, g_n)$  be an  $n$ -tuple from  $G$ . Let  $\tau'(x, \bar{g})$  be a computable partial type so that  $\tau'(x, \bar{g})$  is finitely satisfiable in  $G$ . We first extend  $\tau'(x, \bar{g})$  to a complete type  $\tau(x, \bar{g})$  so that  $\tau(x, \bar{g})$  is also finitely satisfiable in  $G$ . We define an intermediate extension  $\tau''(x, \bar{g})$  of  $\tau'(x, \bar{g})$  first.

Set  $G' = \langle \bar{g} \rangle_{\mathbb{Q}}$ . We may assume that  $\bar{g} = (g_1, \dots, g_n)$  is a valuation basis for  $G'$ . Otherwise, we could replace the parameters  $\bar{g}$  by a valuation basis  $\bar{g}'$  by substituting every occurrence of  $g_i$  in  $\tau'(x, \bar{g})$  with its definition over  $\bar{g}'$  in an effective way. Similarly, we may assume that  $0 < g_1 < g_2 < \dots < g_n$ .

Let  $h_1, \dots, h_l \in \{g_1, \dots, g_n\}$  satisfy

- (a)  $0 < h_1 < h_2 < \dots < h_l$ , and
- (b) for each  $g_i$  with  $1 \leq i \leq n$ , there is exactly one  $j_i$  with  $1 \leq j_i \leq l$  such that  $g_i \in [h_{j_i}]$ .

Let  $r_{j_i} = \frac{g_i}{h_{j_i}}$ . By assumption (ii),  $r_{j_i} \in A_{[h_{j_i}]} = S$ . Since  $S$  is a Scott set, there is some  $r' \in S$  (e.g.,  $r' = r_{j_1} \oplus \dots \oplus r_{j_n}$ , the *join* of the  $r_{j_i}$ ) that computes each of the  $r_{j_i}$ .

Let  $\tau''(x, \bar{g})$  be a partial type that contains all formulas in  $\tau'(x, \bar{g})$  as well as the formulas described below (written in terms of the appropriate parameters in  $\bar{g}$ ).

- (a') For all  $i$  satisfying  $1 \leq i < l$  and  $n \in \omega$ , include the formula  $0 < h_i \wedge nh_i < h_{i+1}$ .
- (b') For all  $q \in \mathbb{Q}$  and all  $i$  satisfying  $1 \leq i \leq n$ , if  $q < r_{j_i}$ , include the formula  $qh_{j_i} < g_i$  in  $\tau''(x, \bar{g})$ . Similarly, if  $q > r_{j_i}$ , include the formula  $g_i < qh_{j_i}$  in  $\tau''(x, \bar{g})$ .

**Claim 4.2.** *A formula holds of  $\bar{g}$  in  $G$  if and only if this formula is in any extension of  $\tau''(x, \bar{g})$ . Moreover,  $\tau''(x, \bar{g})$  is computable from  $r'$ .*

*Proof.* Since divisible ordered abelian groups admit quantifier elimination and  $\tau'(x, \bar{g})$  is computable, it suffices to show we can deduce the order of any two terms in  $\bar{g}$  from formulas in  $\tau''(x, \bar{g})$  computably from  $r'$ . Note that the formulas added to  $\tau'(x, \bar{g})$  by condition (a') are computable from  $r'$  since knowing the order of elements  $h_1, \dots, h_l$  is only finite information. Consider a linear combination  $s_1g_1 + \dots + s_ng_n$  where  $s_1, \dots, s_n \in \mathbb{Z}$ . Ordering any two non-equal terms in  $\bar{g}$  is the same as determining whether such a non-trivial linear combination is positive or negative. Suppose  $i_k$  is the largest index in the linear combination for which  $s_{i_k} \neq 0$ . Let  $h = h_{j_{i_k}}$ . To determine whether a nonzero term is positive or negative, we simply need to determine whether the sum of all monomials in this term with non-zero  $s_i$  and valuation  $h$  is positive or negative. Suppose  $s_{i_1}g_{i_1} + \dots + s_{i_k}g_{i_k}$  is this sum. Since  $\bar{g}$  is a valuation basis, this new linear combination is nonzero and is positive if and only if  $s_{i_1}r_{j_{i_1}} + \dots + s_{i_k}r_{j_{i_k}} > 0$  (See [H], Propositions 12 and 13). Hence, we can compute from  $r'$  the ordering between any two terms in  $\bar{g}$ .  $\square$

Since  $\tau'(x, \bar{g})$  is finitely satisfiable in  $G$ , the claim guarantees that  $\tau''(x, \bar{g})$  is finitely satisfiable in  $G$  as well as computable in  $r'$ . Hence, there is an  $r'$ -computable infinite tree  $\mathcal{T}$  such that any path through  $\mathcal{T}$  encodes a complete consistent type  $\tau(x, \bar{g})$  extending  $\tau''(x, \bar{g})$ . Since  $S$  is a Scott set and  $\mathcal{T}$  is computable in  $r' \in S$ , there is some  $r \in S$  such that  $r$  computes a complete extension  $\tau(x, \bar{g})$  of  $\tau''(x, \bar{g})$ .

Recall that  $G' = \langle \bar{g} \rangle_{\mathbb{Q}}$ , and let  $\Gamma'$  be the value set for  $G'$ . We let

$$\begin{aligned} B &= \{b \in G' \mid \tau(x, \bar{g}) \vdash b \leq x\} \\ C &= \{c \in G' \mid \tau(x, \bar{g}) \vdash x \leq c\} \end{aligned}$$

By quantifier elimination for divisible ordered abelian groups, to realize the type  $\tau(x, \bar{g})$ , it suffices to realize the partial type (also computable in  $r \in S$ )

$$(3) \quad \{b \leq x \mid b \in B\} \cup \{x \leq c \mid c \in C\}.$$

If  $\tau(x, \bar{g}) \vdash x = b$  for any  $b \in B$ , then the type in (3) is realized by  $b \in B \subset G$ , and similarly for  $x = c$  with  $c \in C$ , so suppose there are no such equalities. Let  $G'' \succ G$  be such that there is some  $x_0 \in G''$  such that  $G'' \models \tau(x_0, \bar{g})$ . Consider the set  $\Delta = \{v(d - x_0) \mid d \in G'\}$ . We examine three cases regarding the structure of  $\Delta$ .

**Case 1 (Immediate Transcendental)** -  $\Delta$  has no largest element.

We observe that this case does not occur in our context as  $\langle \bar{g}, x_0 \rangle_{\mathbb{Q}} \supseteq G'$  has finite rank. Thus,  $\Delta$  is finite and has a maximum element.

For the remaining two cases, we fix  $d_0 \in G'$  such that  $v(d_0 - x_0)$  is the maximum of  $\Delta$ . We suppose that  $d_0 \in B$ . The argument in the

case that  $d_0 \in C$  is symmetric. Consider the partial type (which is also computable in  $r \in S$ )

$$(4) \quad \{b - d_0 < x' \mid b \in B\} \cup \{x' < c - d_0 \mid c \in C\}.$$

It is clear that if  $x'$  satisfies this cut, then  $x' + d_0$  satisfies (3). We show that this cut is realized in  $G$  in the remaining two cases.

**Case 2** (*Residue Transcendental*) -  $\Delta$  has a largest element, which is in  $\Gamma'$

We can show as in the proof of Theorem 3.1 in [Ku90] the following claim.

**Claim 4.3.** *There exist  $b_0 \in B$  and  $c_0 \in C$  such that for all  $b \in B$  and  $c \in C$  with  $b_0 \leq b$  and  $c \leq c_0$ ,*

$$\begin{aligned} v(b - d_0) &= v(x_0 - d_0) = v(c - d_0) \text{ and, hence,} \\ v(b - x_0) &= v(x_0 - d_0) = v(c - x_0). \end{aligned}$$

By the claim, we have that for all  $b \in B$  and  $c \in C$  with  $b \geq b_0$  and  $c \leq c_0$ ,

$$\frac{b - d_0}{b_0 - d_0} < \frac{x_0 - d_0}{b_0 - d_0} < \frac{c - d_0}{b_0 - d_0}.$$

Hence, the following partial type (also computable in  $r \in S$ )

$$(5) \quad \left\{ \frac{b - d_0}{b_0 - d_0} < x' \mid b \in B \ \& \ b \geq b_0 \right\} \cup \left\{ x' < \frac{c - d_0}{b_0 - d_0} \mid c \in C \ \& \ c \leq c_0 \right\}$$

is a cut of  $\mathbb{R}$  filled by some  $\hat{r} \in \mathbb{R}$ . Since  $r \in S$  computes the cut for  $\hat{r}$ , we have that  $\hat{r} \in S$ . Thus, by assumption (ii), there is some  $\hat{g} \in G$  such that  $\frac{\hat{g}}{b_0 - d_0} = \hat{r}$  (since  $\hat{r} \in S = A_{[b_0 - d_0]}$ ). Since  $\hat{r}$  fills the cut described in (5), by definition of  $A_{[b_0 - d_0]}$ , the element  $\hat{g}$  fills the cut described in (4), as desired.

**Case 3** (*Group Transcendental*) -  $\Delta$  has a largest element, which is not in  $\Gamma'$

Consider the sets

$$\Delta_1 = \{v(c - d_0) \mid c \in C\} \text{ and } \Delta_2 = \{v(b - d_0) \mid b \in B \ \& \ b > d_0\}.$$

As in the proof of Theorem 3.1 in [Ku90], we can show the following.

**Claim 4.4.**  $\Delta_1 < v(d_0 - x_0) < \Delta_2$ .

Since  $G'$  has finite rank,  $\Delta_1$  and  $\Delta_2$  are finite and form a cut in the value set  $\Gamma$ . By (i),  $\Gamma$  is a dense linear ordering without endpoints, so there is some  $y \in G$  with  $y > 0$  that fills this cut in  $\Gamma$ . Then, for all  $c \in C$  and  $b \in B$  with  $b > d_0$  we have  $v(b - d_0) > v(y) > v(c - d_0)$  so  $b - d_0 < y < c - d_0$ . Hence  $y$  fills the cut given in (4), finishing the case where  $\Delta$  has a largest element, which is not an element of  $\Gamma'$ . This completes the proof that the properties stated are sufficient to guarantee that  $G$  is recursively saturated.  $\square$



## 5. RECURSIVELY SATURATED REAL CLOSED FIELDS

**Definition 5.1.** Let  $r \in \mathbb{R}$ . Let  $R$  be a real closed field. Let  $\bar{d}$  be a tuple of parameters in  $R$ . We say a length  $\omega$  sequence of elements  $(a_i)_{i < \omega} \subset RC(\mathbb{Q}(\bar{d}))$  is *computable in  $r$*  if there is an  $r$ -computable sequence of formulas  $(\theta_i(x, \bar{d}))_{i < \omega}$  such that  $\theta_i(x, \bar{d})$  defines  $a_i$  in  $R$  for all  $i < \omega$ .

**Theorem 5.2.** Let  $R$  be a real closed field,  $v$  its natural valuation,  $G$  its value group and  $k$  its residue field. Then,  $R$  is recursively saturated in the language of ordered fields if and only if there is a Scott set  $S$  such that

- (i)  $G$  is recursively saturated with archimedean components all equal to  $S$ ,
- (ii)  $(k, +, \cdot, 0, 1, <) \cong (S, +, \cdot, 0, 1, <)$ ,
- (iii) every pseudo Cauchy sequence of length  $\omega$  in a subfield of  $R$  of finite absolute transcendence degree over  $\mathbb{Q}$  that is computable in an element of  $S$  has a pseudo limit in  $R$ .
- (iv) every type realized by some  $n$ -tuple  $\bar{a}$  in  $R$  is computable in an element of  $S$ .

*Proof.* We first suppose that  $R$  is recursively saturated. We show that there is a Scott set  $S$  such that conditions (i), (ii), (iii), and (iv) hold with this  $S$ .

(ii) Since  $R$  is recursively saturated as an ordered field,  $(R, +, 0, <)$  is recursively saturated as a divisible ordered abelian group. By Theorem 4.1, there is some Scott set  $S$  such that the archimedean components  $A_{[r],r}$  of  $(R, +, 0, <)$  equal  $S$  for all nonzero  $r \in R$ . In particular, we have that  $A_{[1],1} = S$ . Hence,  $(k, +, 0, <) \cong (S, +, 0, <)$ . Since  $R$  is a real closed field,  $k$  is a real closed field as well. Hence, there is a subset  $K \subset \mathbb{R}$  that is a real closed field isomorphic to  $k$  and an isomorphism  $\phi$  from  $(S, +, 0, <)$  to  $(K, +, 0, <)$ . By Hölder's Theorem,  $\phi(x) = rx$  for some  $r \in \mathbb{R}$ . We show that there is a field isomorphism from  $S$  to  $K$ . Since  $S$  is a Scott set,  $S$  is a real closed subfield of  $\mathbb{R}$ . Since  $1 \in S \cap K$ , we have  $r \in K$  and  $\frac{1}{r} \in S$ . Since  $S$  and  $K$  are in fact sets of reals that form fields,  $r, \frac{1}{r} \in S \cap K$ . Hence,  $S = K$  (given  $s \in S$ ,  $\frac{s}{r} \in S$  so  $\phi(\frac{s}{r}) = s \in K$ , and the other containment is similar). So, the identity function from  $S$  to  $K$  is a *field* isomorphism, giving the desired result.

(iv) Let  $\gamma(\bar{x})$  be a type realized by the tuple  $\bar{a}$  in  $R$ . Let  $r \in \mathbb{R}$  have the same Turing degree as  $\gamma$ , and let  $\Psi$  be the Turing reduction computing  $r$  from  $\gamma$ . It suffices to show  $r \in S$ . Let  $(\theta_i(\bar{x}))_{i \in \omega}$  be a fixed effective enumeration of all formulas in the language of ordered fields. Given  $\sigma \in 2^{<\omega}$ , let  $\theta_\sigma(\bar{x})$  denote the conjunction of the formulas  $\theta_i(\bar{x})$  such that  $\sigma(i) = 1$  and the formulas  $\neg\theta_i(\bar{x})$  such that  $\sigma(i) = 0$ .

We enumerate the formula

$$\theta_\sigma(\bar{a}) \rightarrow q < x < q'$$

into the partial type  $\tilde{\gamma}(\bar{a}, x)$  if  $\Psi$  computes that its output real must be between  $q$  and  $q'$  for  $q < q' \in \mathbb{Q}$  from  $\sigma$ . The partial type  $\tilde{\gamma}(\bar{a}, x)$  is computably enumerable and finitely satisfiable. Since  $R$  is recursively saturated, there is some  $\tilde{r}$  so that  $\tilde{\gamma}(\bar{a}, \tilde{r})$  holds in  $R$ . Since  $\gamma(\bar{a})$  holds in  $R$  and  $\Psi$  computes  $r$  from  $\gamma$ , we have  $r = \tilde{r} \in A_{[1],1} = S$ .

(i) We first show that all the archimedean components of  $G$  equal  $S$ . Let  $r \in S$ . Since  $S = A_{[1],1}$  where  $A_{[1],1}$  is an archimedean component of  $(R, +, 0, <)$ , there is some  $a \in R$  such that  $r = \frac{a}{1} \in A_{[1],1}$ . Let  $g$  be a nonzero element of  $G$  so  $v(a_g) = g$  for some  $a_g > 0$  in  $R$ . We show  $r \in A_{[g],g}$ . Note that  $A_{[g],g}$  is an archimedean component of  $(G, +, 0, <)$ . The group  $(G, +, 0, <)$  is isomorphic to a section of the *multiplicative* group  $(R^{>0}, \cdot, 1, <)$  as opposed to the additive group  $(R, +, 0, <)$ . If  $r \in \mathbb{Q}$ , then  $r \in A_{[g],g}$  since  $G$  is divisible, so we may suppose  $r \notin \mathbb{Q}$ .

Let  $\delta(x, a, a_g)$  be the partial type in the language of real closed fields consisting of all formulas with  $q, q' \in \mathbb{Q}^{>0}$  and  $q < q'$  of the form

$$q < a < q' \rightarrow a_g^q < x < a_g^{q'}$$

The set of formulas  $\delta(x, a, a_g)$  is computable and finitely satisfiable in  $R$  since  $R$  is real closed. Since  $R$  is recursively saturated, there is some  $a' \in R$  so that  $\delta(a', a, a_g)$  holds in  $R$ . Let  $g' = v(a') \in G$ . Then,  $\frac{g'}{g} = r$ , so  $r \in A_{[g'],g'}$ .

Now, let  $r \in A_{[g],g}$ , so there is some  $g' \in G$  such that  $r = \frac{g'}{g}$ . Let  $a_{g'}, a_g \in R$  be positive elements such that  $v(a_{g'}) = g'$  and  $v(a_g) = g$ . If  $r \in \mathbb{Q}$ , then it is clear  $r \in S$  as all rationals are computable. Otherwise, let  $\delta'(x, a_{g'}, a_g)$  be the partial type in the language of real closed fields consisting of all formulas with  $q, q' \in \mathbb{Q}^{>0}$  and  $q < q'$  of the form

$$a_g^q < a_{g'} < a_g^{q'} \rightarrow q < x < q'$$

As before, the set of formulas  $\delta'(x, a_{g'}, a_g)$  is computable and finitely satisfiable in  $R$ , so there is some  $a \in R$  so that  $\delta'(a, a_{g'}, a_g)$  holds in  $R$ . Then,  $r = \frac{a}{1} \in k = S$ . Hence, we have  $A_{[g],g} = S = k$  for all  $g \in G$ , so we can simply refer to  $A_{[g]}$  instead of  $A_{[g],g}$ .

We now show that  $G$  is recursively saturated. Let  $\bar{g} = (g_1, \dots, g_n)$  be an  $n$ -tuple from  $G$ . Let  $\beta'(x, \bar{g})$  be a computable partial type in the language of ordered abelian groups so that  $\beta'(x, \bar{g})$  is finitely satisfiable in  $G$ . By the argument found at the beginning of the sufficiency proof for Theorem 4.1, we can find an  $r \in S$  such that  $r$  computes a complete extension  $\beta(x, \bar{g})$  of  $\beta'(x, \bar{g})$  that is finitely satisfiable in  $G$ . Since  $r \in S = k$ , there exists some  $a \in R$  such that  $\frac{a}{1} = r$ .

Set  $G' = \langle \bar{g} \rangle_{\mathbb{Q}}$ . If  $\beta \vdash x = g$  for any  $g \in G'$ , then we are done. Otherwise, let

$$\begin{aligned} B &= \{b \in G' \mid \tau(x, \bar{g}) \vdash b < x\} \\ C &= \{c \in G' \mid \tau(x, \bar{g}) \vdash x < c\} \end{aligned}$$

As in Theorem 4.1, it suffices to realize the partial type (also computable in  $r \in S$ ) in  $G$

$$(6) \quad \{b < x \mid b \in B\} \cup \{x < c \mid c \in C\}$$

that describes a cut in  $G$ . We translate realizing this cut in  $G$  into realizing a particular partial type (in the language of ordered fields) in  $R$ .

Let  $R' = RC(\mathbb{Q}(\bar{g}))$ . Take  $d_1, \dots, d_n \in R'$  so that  $d_1, \dots, d_n > 0$ ,  $\{v(d_i) \mid 1 \leq i \leq n\}$  is a basis for  $G'$ , and the multiplicative subgroup

$$\left\{ \prod_{i=1}^n d_i^{q_i} \in R' \mid q_i \in \mathbb{Q} \text{ for } 1 \leq i \leq n \right\}$$

is a section for  $G'$  in  $R'$ . We show that there is a computably enumerable partial type in the language of ordered fields  $\tilde{\beta}(x, d_1, \dots, d_n, a)$  (with parameters in  $R$ ) that corresponds to the cut described by  $\beta(x, \bar{g})$  over  $G'$ . Note that

$$B = \left\{ \sum_{i=1}^n q_i v(d_i) \in G' \mid \sum_{i=1}^n q_i v(d_i) < b \text{ \& } b \in B \text{ \& } q_1, \dots, q_n \in \mathbb{Q} \right\} \text{ and}$$

$$C = \left\{ \sum_{i=1}^n q_i v(d_i) \in G' \mid c < \sum_{i=1}^n q_i v(d_i) \text{ \& } c \in C \text{ \& } q_1, \dots, q_n \in \mathbb{Q} \right\}.$$

Given some  $(q_1, \dots, q_n) \in \mathbb{Q}^n$ , the statement that determines whether  $\sum_{i=1}^n q_i v(d_i)$  is in  $B$  or  $C$  can be computably located in an effective listing of all formulas. Since  $r \in S$  computes the complete type  $\beta(x, \bar{g})$ , there is some Turing reduction  $\Upsilon$  that computes from  $r$  whether a given  $(q_1, \dots, q_n) \in \mathbb{Q}^n$  satisfies  $\sum_{i=1}^n q_i v(d_i) \in B$  or  $\sum_{i=1}^n q_i v(d_i) \in C$ .

We now describe  $\tilde{\beta}(x, d_1, \dots, d_n, a)$ . For each pair of rationals  $(q < q')$ , each stage  $s \in \mathbb{N}$ , and  $(q_1, \dots, q_n) \in \mathbb{Q}^n$ , compute whether Turing reduction  $\Upsilon$ , using only the information that some real  $\tilde{r}$  satisfies  $q < \tilde{r} < q'$ , halts in  $s$  steps and outputs whether  $\sum_{i=1}^n q_i v(d_i)$  is in  $B$  or  $C$ . If  $\Upsilon$  halts in this situation, enumerate the formula

$$q < a < q' \rightarrow \prod_{i=1}^n d_i^{q_i} < x \text{ if } \Upsilon \text{ computes that } \sum_{i=1}^n q_i v(d_i) \in C \text{ or}$$

$$q < a < q' \rightarrow x < \prod_{i=1}^n d_i^{q_i} \text{ if } \Upsilon \text{ computes that } \sum_{i=1}^n q_i v(d_i) \in B$$

into  $\tilde{\beta}(x, d_1, \dots, d_n, a)$ .

The partial type  $\tilde{\beta}(x, d_1, \dots, d_n, a)$  is finitely satisfiable in  $R$  because  $R$  is a dense linear ordering without endpoints. Since  $\tilde{\beta}(x, d_1, \dots, d_n, a)$  is a computably enumerable and  $R$  is recursively saturated, there exists some  $d \in R$  so that  $\tilde{\beta}(d, d_1, \dots, d_n, a)$  holds in  $R$ . By our choice of  $a$  and definition of  $\tilde{\beta}$ , we have that  $B < v(d) < C$ . So,  $\beta(v(d), \bar{g})$  holds in  $G$ , as desired.

(iii) Let  $(a_i)_{i < \omega} \subset R'$  be a pseudo Cauchy sequence in a subfield  $R'$  of  $R$  of finite absolute transcendence degree over  $\mathbb{Q}$ . Moreover, suppose that  $(a_i)_{i < \omega}$  is computable in  $r \in S = k$ . By definition, there is an  $r$ -computable sequence of formulas  $(\theta_i(x, \bar{y}))_{i < \omega}$  and tuple  $\bar{d}$  from  $R'$  such that  $\theta_i(x, \bar{d})$  defines  $a_i$  in  $R$ , where  $R' = RC(\mathbb{Q}(\bar{d}))$ . Let  $\Phi$  denote the Turing reduction from  $r$  to this sequence. Since  $r \in S = k$ , there exists some  $a \in R$  such that  $\frac{a}{1} = r$ .

For each pair of rationals  $q < q'$  and any  $i, s \in \mathbb{N}$ , compute whether Turing reduction  $\Phi$ , using only the information that some real  $\tilde{r}$  satisfies  $q < \tilde{r} < q'$ , halts in  $s$  steps and outputs indices for formulas  $\theta_i(x, \bar{y})$  and  $\theta_{i+1}(x, \bar{y})$ . If  $\Phi$  halts in this situation, enumerate the following formula into the partial type  $\kappa(x, \bar{d}, a)$

$$q < a < q' \rightarrow [(\exists z_i, z_{i+1})(\theta_i(z_i, \bar{d}) \wedge \theta_{i+1}(z_{i+1}, \bar{d})) \wedge n|x - z_{i+1}| < |z_i - z_{i+1}|]$$

for each  $n \in \omega$ . Note that  $\kappa(x, \bar{d}, a)$  is computably enumerable and finitely satisfiable in  $R$ . Given a finite set of formulas  $D \subset \kappa(x, \bar{d}, a)$ , let  $j < \omega$  be the largest number such that  $\theta_j(x, \bar{d})$  appears as a subformula of an element in  $D$ . Then,  $a_{j+1} \in R$  satisfies all formulas in  $D$  since  $(a_i)_{i < \omega}$  is pseudo Cauchy. Since  $R$  is recursively saturated, there is some  $\tilde{a} \in R$  such that  $\kappa(\tilde{a}, \bar{d}, a)$  holds in  $R$ . This implies that  $v(\tilde{a} - a_{i+1}) > v(a_{i+1} - a_i)$  for all  $i < \omega$ . From  $v(\tilde{a} - a_i) \geq \min\{v(\tilde{a} - a_{i+1}), v(a_{i+1} - a_i)\}$  it follows that  $v(\tilde{a} - a_i) = v(a_{i+1} - a_i)$  for all  $i < \omega$ . Hence,  $\tilde{a}$  is a pseudo limit of  $(a_i)_{i < \omega}$ , as required, and the four conditions (i), (ii), (iii), and (iv) are necessary if  $R$  is recursively saturated.

Let  $R$  be a real closed field. We assume that there is a Scott set  $S$  for which conditions (i), (ii), (iii), and (iv) hold for  $R$  and  $S$ . We show that  $R$  is recursively saturated.

Let  $\bar{a} = (a_1, \dots, a_n)$  be a finite tuple from  $R$ , and let  $\tau'(x, \bar{a})$  be a computable set of formulas that is finitely satisfiable in  $R$ . We first extend  $\tau'(x, \bar{y})$  to a complete type  $\tau(x, \bar{y})$  so that  $\tau(x, \bar{a})$  is also finitely satisfiable in  $G$ . We first make an intermediate extension  $\tau''(x, \bar{y})$  of  $\tau'(x, \bar{y})$ . Let  $\gamma(\bar{y})$  be the complete type of  $\bar{a}$  in  $R$ . We then let  $\tau''(x, \bar{y}) = \tau'(x, \bar{y}) \cup \gamma(\bar{y})$ . By condition (iv), the type  $\gamma(\bar{y})$  is computable in some  $r' \in S$ , so  $\tau''(x, \bar{y})$  is as well. Moreover,  $\tau''(x, \bar{a})$  is finitely satisfiable in  $R$ . Hence, there is an  $r'$ -computable infinite tree  $\mathcal{T}$  such that any path through  $\mathcal{T}$  encodes a complete consistent type extending  $\tau''(x, \bar{a})$ . Since  $S$  is a Scott set and  $\mathcal{T}$  is computable in  $r' \in S$ , there is some  $r \in S$  such that  $r$  computes a complete extension  $\tau(x, \bar{a})$  of  $\tau''(x, \bar{a})$ .

Set  $R' = RC(\mathbb{Q}(\bar{a}))$ . We set

$$B = \{b \in R' \mid \tau(x, \bar{a}) \vdash b \leq x\} \text{ and} \\ C = \{c \in R' \mid \tau(x, \bar{a}) \vdash x \leq c\}.$$

Real closed fields, like divisible ordered abelian groups, have quantifier elimination. Hence, to realize the type  $\tau(x, \bar{a})$ , it suffices to realize the

partial type (also computable in  $r \in S$ )

$$(7) \quad \{b \leq x \mid b \in B\} \cup \{x \leq c \mid c \in C\}.$$

If  $\tau(x, \bar{a}) \vdash x = b$  for any  $b \in B$ , then the type in (7) is realized by  $b \in B \subset R$ , and similarly for  $x = c$  with  $c \in C$ , so suppose there are no such equalities. Let  $R'' \succ R$  be such that there is some  $x_0 \in R''$  satisfying  $R'' \models \tau(x_0, \bar{a})$ . Consider the set  $\Delta = \{v(d - x_0) \mid d \in R'\}$ . We examine three cases for the structure of  $\Delta$ , as we did in Theorem 4.1 in the group case.

**Case 1** (*Immediate Transcendental*) -  $\Delta$  has no largest element. In this case, for all  $d \in R'$  there is a  $d' \in R'$  such that  $v(d - x_0) < v(d' - x_0)$ . We construct a pseudo Cauchy sequence  $(a_i)_{i < \omega}$  that is computable in some element of  $S$  and has a pseudo limit  $a \in R$  satisfying  $B < a < C$ . By effective quantifier elimination for real closed fields, there is a computable enumeration of formulas  $\{\psi_i(x, \bar{a})\}_{i < \omega}$  such that

- (a) every element in  $R'$  is defined by exactly one formula in this sequence and
- (b) if  $a_i$  and  $a_j$  are defined by  $\psi_i(x, \bar{a})$  and  $\psi_j(x, \bar{a})$  respectively, then determining whether  $a_i < a_j$  and whether  $a_i \in B$  or  $a_i \in C$  in  $R'$  is  $r$ -computable.

Let  $a_i$  denote the element in  $R'$  that satisfies the definition  $\psi_i(a_i, \bar{a})$ . We define a tree  $\mathcal{T} \subset 2^{<\omega}$  computable in  $r$ . For any  $\sigma \in 2^{<\omega}$ , we put  $\sigma \in \mathcal{T}$  if the following two properties hold.

- (I) For all  $i < \text{length}(\sigma)$ , set  $a'$  equal to 0 if for all  $j \leq i$ ,  $\sigma(j) = 0$ , and otherwise, set  $a'$  equal to  $a_{j'}$  where  $j' = \max\{j \leq i \mid \sigma(j) = 1\}$ . Then,

$$(\forall j \leq i)(a_j \in B \implies a_j \leq a') \ \& \ (a_j \in C \implies a' \leq a_j))$$

- (II)  $(\forall i < j < k < n = \text{length}(\sigma))$   
 $(\sigma(i) = \sigma(j) = \sigma(k) = 1 \implies n|a_k - a_j| < |a_j - a_i|)$

It is clear  $\mathcal{T}$  is a tree by definition. We now show  $\mathcal{T}$  is infinite. Since  $\Delta$  has no largest element, there exists a cofinal sequence in  $\Delta$ . Moreover, since  $R'$  is countable and  $B < x_0 < C$  in  $R''$ , we can take this cofinal sequence to have the form  $(v(a_{i_l} - x_0))_{l < \omega}$  and to satisfy the following two properties.

- (a) The sequences  $(i_l)_{l < \omega}$  and  $(v(a_{i_l} - x_0))_{l < \omega}$  are increasing.
- (b) For each  $n < \omega$ , if we set  $a'$  equal to 0 if no  $j \leq n$  equals some  $i_l$  and we set  $a'$  equal to  $a_{i_{l'}}$  where index  $i_{l'} = \max\{i_l \leq n\}$  otherwise, then

$$(\forall j \leq n)(a_j \in B \implies a_j \leq a') \ \& \ (a_j \in C \implies a' \leq a_j))$$

Let  $\mathcal{P}' \in 2^\omega$  be defined so that  $\mathcal{P}'(j) = 1$  if and only if  $j = i_l$  for some  $l \in \omega$ . We show that  $\mathcal{P}'$  is a path through  $\mathcal{T}$ , so  $\mathcal{T}$  is infinite. Let  $\sigma_n = \mathcal{P}' \upharpoonright n$ . It is clear that  $\sigma_n$  satisfies (I) by definition. We show

that  $\sigma_n$  satisfies (II). Suppose  $i < j < k < n$  with

$$\sigma_n(i) = \sigma_n(j) = \sigma_n(k) = 1,$$

i.e.,  $i = i_l$ ,  $j = i_{l'}$  and  $k = i_{l''}$  with  $l < l' < l''$ . It suffices to show that  $v(a_j - a_i) < v(a_k - a_j)$ . We have that

$$v(a_{i_l} - x_0) < v(a_{i_{l'}} - x_0) < v(a_{i_{l''}} - x_0)$$

and

$$\begin{aligned} v(a_j - a_i) &= \min(v(a_{i_{l''}} - x_0), v(a_{i_l} - x_0)) = v(a_{i_l} - x_0) \\ v(a_k - a_j) &= \min(v(a_{i_{l''}} - x_0), v(a_{i_{l'}} - x_0)) = v(a_{i_{l'}} - x_0) \text{ so} \\ v(a_j - a_i) &< v(a_k - a_j) \text{ as desired.} \end{aligned}$$

Hence,  $\mathcal{T}$  is an infinite tree computable in  $r$ . Since  $S$  is a Scott set, there exists a path  $\mathcal{P}$  through  $\mathcal{T}$  computable in some  $t \in S$ . Since  $B$  and  $C$  form a proper cut in  $R'$ , there are infinitely many  $j < \omega$  such that  $\mathcal{P}(j) = 1$  by property (I) of  $\mathcal{T}$ . We then can compute in  $t$ , for each  $l < \omega$ , the index  $k_l$  such that  $\mathcal{P}(k_l) = 1$  and  $|\{j \leq k_l \mid \mathcal{P}(j) = 1\}| = k_l$ . By property (II) of the definition of  $\mathcal{T}$ , the sequence  $(a_{k_l})_{l < \omega}$  is pseudo Cauchy. Since  $(\psi_{k_l}(x, \bar{y}))_{l < \omega}$  is computable in  $t$ , the sequence  $(a_{k_l})_{l < \omega}$  (defined by this sequence of formulas over  $\bar{a}$ ) has a pseudo limit  $a \in R$  by assumption (iii).

We show that  $B < a < C$  holds in  $R$ , and so  $a$  realizes the type in (7). Let  $b \in B \subset R'$ . We claim that  $b < a$ . Otherwise,  $a \leq b < x_0$ . By definition of  $\mathcal{T}$ , there exists some  $l < \omega$  such that  $a \leq b \leq a_{k_l} < a_{k_{l+1}}$ . Then,  $v(a - a_{k_{l+1}}) \leq v(a - a_{k_l})$ . Since  $a$  is a pseudo limit for  $(a_{k_l})_{l < \omega}$ ,

$$v(a - a_{k_{l+1}}) = v(a_{k_{l+2}} - a_{k_{l+1}}) > v(a_{k_{l+1}} - a_{k_l}) = v(a - a_{k_l}),$$

a contradiction, so we have shown  $b < a$ . The argument that  $a < c$  for any  $c \in C$  is similar.

**Case 2** (*Residue Transcendental*) -  $\Delta$  has a largest element  $g \in v(R')$

Assume that  $\Delta$  has a largest element  $g \in v(R')$ . Let  $a > 0$  be such that  $a, d_0 \in R'$  and  $v(d_0 - x_0) = g = v(a)$ .

**Claim 5.3.** *There exists  $b_0 \in B$  and  $c_0 \in C$  such that for all  $b \in B$  with  $b \geq b_0$  and for all  $c \in C$  with  $c \leq c_0$ , we have*

$$\begin{aligned} v(b - d_0) &= g = v(a) = v(c - d_0) \text{ and, hence,} \\ v(b - x_0) &= g = v(a) = v(x_0 - d_0) = v(c - x_0). \end{aligned}$$

Like the corresponding Claim 4.3, its proof is a straightforward adaptation of the proof of the analogous statement in Theorem 3.1 in [Ku90].

Consider the partial type (also computable in  $r$ ):

$$(8) \quad \left\{ \frac{b - d_0}{a} < x \mid b \in B \ \& \ b \geq b_0 \right\} \cup \left\{ x < \frac{c - d_0}{a} \mid c \in C \ \& \ c \leq c_0 \right\}$$

If some  $x'$  realizes the type in (8) then  $x = a \cdot x' + d_0$  realizes the type in (7). So, it suffices to find such an  $x' \in R$ . Again, suppose  $d_0 \in B$ , as the case where  $d_0 \in C$  is symmetric.

By the claim, for all  $b \in B$  and  $c \in C$  with  $b_0 \leq b < c \leq c_0$

$$v\left(\frac{b-d_0}{a}\right) = v\left(\frac{x_0-d_0}{a}\right) = v\left(\frac{c-d_0}{a}\right) = 0$$

Furthermore, if we take the residues of these elements, we have

$$\frac{\overline{b-d_0}}{a} < \frac{\overline{x_0-d_0}}{a} < \frac{\overline{c-d_0}}{a}.$$

All inequalities in the line above are strict since otherwise  $g = v(x_0-d_0)$  is not the maximum of  $\Delta$ . Hence, the two sets

$$\{q \in \mathbb{Q} \mid q < \frac{b-d_0}{a} \text{ \& } b \in B \text{ \& } b \geq b_0\}$$

$$\{q \in \mathbb{Q} \mid \frac{c-d_0}{a} < q \text{ \& } c \in C \text{ \& } c \leq c_0\}$$

form a cut in  $\mathcal{R}$  that is computable in  $r$ . Let  $r' \in \mathbb{R}$  fill this cut. Since  $r'$  is computable in  $r$ , we have  $r' \in S \cong k$  by assumption (ii). Thus, there is some  $x' \in R$  that realizes the partial type in (8), as desired.

**Case 3** (*Group Transcendental*) -  $\Delta$  has a largest element  $g \notin v(R')$

Let  $d_0 \in R'$  such that  $v(d_0 - x_0) = g$  is the maximum of  $\Delta$ . We suppose that  $d_0 \in B$ ; the case that  $d_0 \in C$  is similar.

Consider the sets

$$\Delta_1 = \{v(c-d_0) \mid c \in C\} \text{ and } \Delta_2 = \{v(b-d_0) \mid b \in B \text{ \& } b > d_0\}.$$

**Claim 5.4.**  $\Delta_1 < g < \Delta_2$ .

As for the corresponding Claim 4.4 in the group case, the proof of the above claim can be found in Theorem 3.1 in [Ku90].

By the claim,

$$\eta(y) = \{v(c-d_0) < y \mid c \in C\} \cup \{y < v(b-d_0) \mid b \in B \text{ \& } b > d_0\}$$

is a type in  $G$  with parameters in  $G' = v(R')$  that describes a cut in  $G$ . The real closed field  $R'$  has finite absolute transcendence degree, so  $G'$  has finite rational rank (see [SZ], Section 10). Take  $d_1, \dots, d_n \in R'$  so that  $\{v(d_i) \mid 1 \leq i \leq n\}$  is a basis for  $G'$  and the multiplicative subgroup

$$\left\{ \prod_{i=1}^n d_i^{q_i} \in R' \mid q_i \in \mathbb{Q} \text{ for } 1 \leq i \leq n \right\}$$

is a section for  $G'$  in  $R'$ . We show that there is a computably enumerable partial type  $\tilde{\eta}(y, v(d_1), \dots, v(d_n), h)$  (with parameters in  $G$ ) that describes the same cut over  $G'$  as  $\eta(y)$ .

Note that

$$\Delta_1 = \left\{ \sum_{i=1}^n q_i v(d_i) \in G' \mid \prod_{i=1}^n d_i^{q_i} < c-d_0 \text{ \& } c \in C \text{ \& } q_i \in \mathbb{Q} \right\} \text{ and}$$

$$\Delta_2 = \left\{ \sum_{i=1}^n q_i v(d_i) \in G' \mid b-d_0 < \prod_{i=1}^n d_i^{q_i} \text{ \& } b \in B \text{ \& } b > d_0 \text{ \& } q_i \in \mathbb{Q} \right\}.$$

Recall that  $r \in S$  computes the complete type  $\tau(x, \bar{a})$  extending the computable partial type  $\tau'(x, \bar{a})$  we wish to realize in  $R$ . Moreover, given some  $(q_1, \dots, q_n) \in \mathbb{Q}^n$ , the statement that determines whether  $\sum_{i=1}^n q_i v(d_i)$  is in  $\Delta_1$  or  $\Delta_2$  can be computably located in  $\tau$ . Hence, there is some Turing reduction  $\Upsilon$  that computes from  $r$  whether a given  $(q_1, \dots, q_n) \in \mathbb{Q}^n$  satisfies  $\sum_{i=1}^n q_i v(d_i) \in \Delta_1$  or  $\sum_{i=1}^n q_i v(d_i) \in \Delta_2$ .

Take a nonzero  $g \in v(R)$ ; such a  $g$  exists by (i) and Theorem 4.1. Since  $r \in S = A_{[g],g}$  by assumption (i), there exists some  $g_r \in v(R)$  such that  $\frac{g_r}{g} = r$ .

We now describe  $\tilde{\eta}(y, v(d_1), \dots, v(d_n), g_r)$ . For each pair of rationals  $(q < q')$ , each stage  $s \in \mathbb{N}$ , and  $(q_1, \dots, q_n) \in \mathbb{Q}^n$ , compute whether Turing reduction  $\Upsilon$ , using only the information that some real  $\tilde{r}$  satisfies  $q < \tilde{r} < q'$ , halts in  $s$  steps and outputs whether  $\sum_{i=1}^n q_i v(d_i)$  is in  $\Delta_1$  or  $\Delta_2$ . If  $\Upsilon$  halts in this situation, enumerate either the formula

$$\begin{aligned} qg < g_r < q'g \rightarrow \sum_{i=1}^n q_i v(d_i) < y \text{ if computation says } \sum_{i=1}^n q_i v(d_i) \in \Delta_1 \\ & \text{or the formula} \\ qg < g_r < q'g \rightarrow y < \sum_{i=1}^n q_i v(d_i) \text{ if computation says } \sum_{i=1}^n q_i v(d_i) \in \Delta_2 \end{aligned}$$

into  $\tilde{\eta}(y, v(d_1), \dots, v(d_n), g_r)$ . The partial type  $\tilde{\eta}(y, v(d_1), \dots, v(d_n), g_r)$  is computably enumerable, and it is finitely satisfiable in  $G$  because  $G$  is a divisible group. By assumption (i),  $G$  is recursively saturated and this implies there exists some  $h \in G$  so that  $\tilde{\eta}(h, v(d_1), \dots, v(d_n), g_r)$  holds in  $G$ . By our choice of  $g_r$  and definition of  $\tilde{\eta}$ , we have that  $\Delta_1 < h < \Delta_2$ .

Let  $a_h \in R$  satisfy  $v(a_h) = h$  and  $a > 0$ . Then,

$$(\forall c \in C)(\forall b \in B)(b > d_0 \implies v(c - d_0) < v(a) < v(b - d_0))$$

$$\text{and, hence, } (\forall c \in C)(\forall b \in B)(b > d_0 \implies b - d_0 < a < c - d_0).$$

Thus,  $B < a + d_0 < C$ , so  $a + d_0$  realizes the type given in (7), as required.

Therefore, in each of the three cases, we satisfied the type  $\tau$ . Hence,  $R$  is recursively saturated. This completes the sufficiency direction of the proof.  $\square$

It is unclear whether condition (iv) follows from the other three conditions listed in Theorem 5.2. In Theorem 4.1, we used a valuation basis for  $G$  to avoid the need for such a condition. However, a real closed field of finite absolute transcendence degree need not admit a valuation transcendence basis; see [Kuh] for a precise definition of valuation transcendence basis and counterexamples.

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